Every planar graph is 4-colourable and 5-choosable – a joint proof

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Abstract

A new straightforward proof of the 4-colour theorem and the 5-choosable theorem is presented which does not require the use of computer programs. The essential idea of the proof is a simple partitioning (case distinction) not considered before, which is based on combined construction and colouring rules for triangulated planar graphs. It turns out that the incorrect part of Kempe's approach [1] to prove the 4-colour theorem is not necessary for the proof.

Keywords: planar graph, triangulation, vertex colouring, chromatic number, list-chromatic number, 4-colourable, 5-choosable.

1 Introduction

A 5-colour theorem (5CT) was proved by Heawood [2] in 1890 in response to Kempe's incorrect proof of the 4-colour theorem (4CT) [1], and 4CT was first proved by Appel and Haken (see e.g. [3]) in 1976 with the help of extensive computer calculations. An improved and independent version of this type of proof was given by Robertson et al. [4] in 1996. The first proof of the list-colouring theorem – every planar graph is 5-choosable – was provided by Thomassen [5] in 1994.

Proofs involving the extensive use of computer programs still seem indigestible to some. Although the author does not share this attitude, proofs should be preferred which can be checked "by hand". Here we present a new joint proof of both theorems which fulfils this requirement.

2 Theorems for colouring and list-colouring planar graphs

Theorem (4CT): *The chromatic number of a planar graph is not greater than 4.*

Theorem (5LCT): *The list-chromatic number of a planar graph is not greater than 5.*

We will work in the vertex-colouring context with the usual assumptions, i.e. a coloured map in the plane or on a sphere is represented by its dual graph with coloured vertices. When two vertices v_k and v_l are connected by an edge $e_{kl} = v_k v_l$ (i.e. the countries on the map have a common line-shaped border), a (proper) colouring requires that their colours $c_k = c(v_k)$ and c_l be different. For convenience, integers denote colours from colour lists *S*, i.e. $c_i \in S(v_i) = \{1, 2, 3, ...\}$ for any vertex v_i. For the proof of the 4-colour theorem, we set $S(v_i) = S$. In the figures, vertex numbers and colours are assigned clockwise.

Without loss of generality the proof can be restricted to (plane) triangulations. A plane graph *G* is called a triangulation if it is connected, without loops, and every region including the infinite region is a triangle (if the infinite region has more than 3 edges, the graph is called near-triangular). It follows from Euler's polyhedral formula that a triangulation with $n \geq 3$ vertices has $3n-6$ edges. A region is a triangle if it is incident with exactly 3 edges.

Any (non-triangulated) planar graph *H* can be generated from a triangulation *G* by removing edges and disconnected vertices, i.e. $H \subseteq G$. The removal of edges reduces the number of restrictions for colouring, i.e. the (list-) chromatic number of *H* is not greater than the corresponding number of *G*.

3 Part 1 of the proof

For any triangulated graph with $n \leq 4$ vertices the proof of both theorems is trivial. Now induction with respect to *n* is carried out.

Let $n \geq 4$, and both theorems be true for n vertices. Consider a planar graph with $n+1$ vertices, i.e. one vertex has to be inserted into the existing triangulation. We have to show that 4 colours (resp. colour lists with ≤ 5 elements for the 5-choosable theorem) are still sufficient.

Let $\bar{v}$ be this additional vertex. Now the problem-solving idea is to find a suitable case distinction – namely the answer to the question: *where do we insert?* There are 3 mutually exclusive "target areas" for $\bar{v}$: the interior of a region, on an edge, or on an existing vertex. However, in a planar graph a coincidence with an existing vertex does not increase the number of vertices – this case is to be excluded for planar embeddings, leaving the first two cases to be treated in more detail.

Figures 1 – 4 illustrate this case distinction. The left diagram in each figure describes the state before the insertion of $\bar{v}$ (the broken lines denote the anticipated edge changes). The right diagram shows the state after insertion, with the thick lines representing the added edges.

Case A: Vertex $\bar{v}$ is inserted into the interior of any finite (**Case A1**, Fig. 1) or of the infinite region (which is the exterior region of a triangle; **Case A2**, Fig. 2). As the topological differences between the 2 subcases and the topological variety in Case A2 – which vertex is shifted into the interior region – are irrelevant for the proof, they will not be discussed here. In order to obtain again a triangulation, we have to add 3 edges to the 3 triangle vertices v_1, v_2, v_3, coloured by 3 different colours. Then a fourth colour is available for $\bar{v}$.

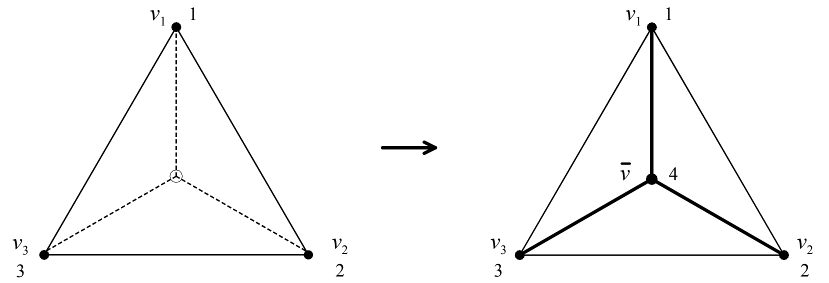

Fig. 1 (Case A1): Vertex $\bar{v}$ is inserted into the interior of a finite region (triangle)

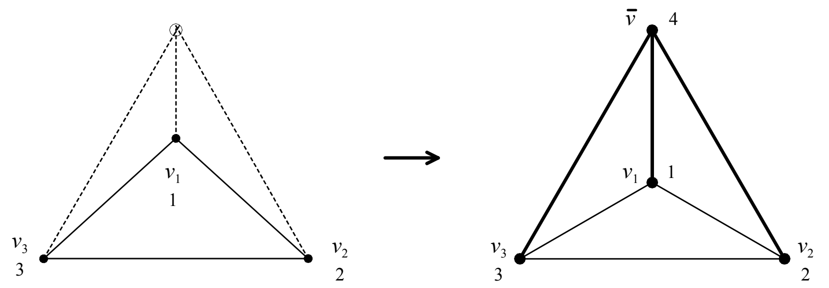

Fig. 2 (Case A2): Vertex $\bar{v}$ is inserted into the interior of the infinite region

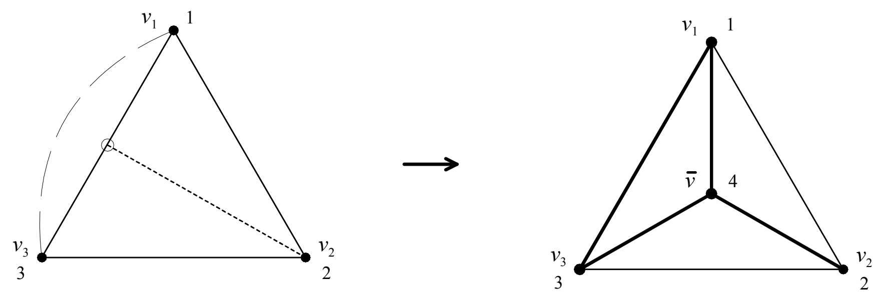

Fig. 3 (Case B1): Vertex $\bar{v}$ is inserted on one edge of the outer circuit B of the graph

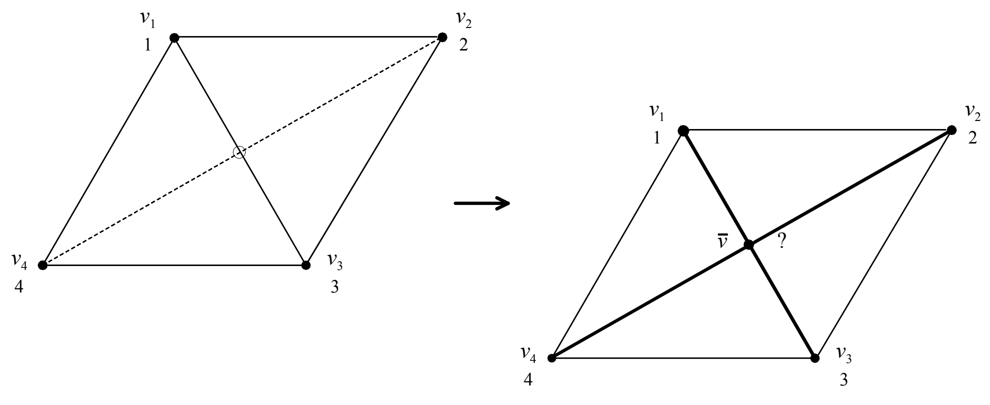

Fig. 4 (Case B2): Vertex $\bar{v}$ is inserted on one edge between 2 finite regions (triangles)

Case B: Vertex $\bar{v}$ is inserted on one edge of a triangle (see Figs. 3 and 4).
Case B1: The selected edge lies on the outer circuit or "hull" $B = \partial G$ of the graph, i.e. it separates one finite from the infinite region (Fig. 3). We remove the "old" edge $v_1 v_3$ and add 3 edges $v_1\bar{v}, v_2\bar{v}, v_3\bar{v}$, together with a "new" edge $v_1 v_3$. As $\bar{v}$ is 3-valent, a fourth colour is available for this vertex. Note that this case is equivalent to Case A1.

Case B2: The selected edge separates two finite regions (Fig. 4). Let $v_1 v_3$ denote this common edge, which is replaced by $v_1\bar{v}$ and $v_3\bar{v}$ (subdivision), and supplemented by edges $v_2\bar{v}$ and $v_4\bar{v}$. Now $\bar{v}$ is 4-valent, and 4 colours are already used for the adjacent vertices $v_1, ..., v_4$ in the worst case. Then a fifth colour can be used for $\bar{v}$.

This concludes the proof that any planar graph is 5-choosable (5LCT). It is known from counter-examples [6] that a list-colouring with less than 5 colours does not exist. With respect to colouring, the 5-colour theorem (5CT) is proved as an intermediate result.

4 Part 2 of the proof (4CT)

In order to proceed with Case B2 of the proof of the 4-colour theorem, we have to show that by rearrangement of colours, one colour can be released and used for the additional vertex $\bar{v}$. This can be simply done by means of Kempe chains [2], in accordance with the correct part of Kempe's proof [1] for the 4-valent vertex case. (Note that Kempe chains are not available in the list-colouring context, as the vertex-specific colour lists may not contain common colours.)

For the completeness of the paper, this part of Kempe's proof is presented anew. Let $H_{i,j}$ denote the subgraph $G \setminus \bar{v}$ which contains all vertices coloured by two different colours $i, j \in \{1, 2, 3, 4\}$. A path $P \subseteq H_{i,j}$ connecting two vertices with colours *i* and *j* is called a Kempe chain.

If $H_{1,3}$ is disconnected in such a way that vertices v_1 and v_3 lie in different components of this subgraph, no Kempe chain P_{13} between v_1 and v_3 exists with alternating colours $\{1, 3\}$. Hence the part of the chain e.g. starting from vertex v_3 can be re-coloured so that $c(v_3) = 1$. Then set $c(\bar{v}) = 3$.

On the other hand, if this Kempe chain P_{13} does exist, the existence of a second Kempe chain P_{24} between vertices v_2 and v_4 with alternating colours $\{2, 4\}$ is excluded by Jordan's curve theorem: Assume that both P_{13} and P_{24} do exist. Vertices v_2 and v_4 lie in regions separated by the Jordan curve $P_{13} \cup v_1 v_3$. The planarity of the graph allows crossings between P_{13} and P_{24} only at a vertex which belongs to both paths, i.e. it has to be coloured with one colour from each subset $\{1, 3\}$ and $\{2, 4\}$ simultaneously, which is impossible. Hence the part of the chain starting e.g. from vertex v_4 can be re-coloured so that $c(v_4) = 2$. Then set $c(\bar{v}) = 4$.

This concludes the proof of Case B2 and of the 4-colour theorem.

5 Conclusions

Note that the essential feature of this proof is the use of simultaneous construction and colouring rules for graphs. This alternative approach does not try to profit from the existence of a 5-valent vertex $\bar{v}$ and from assumptions about colouring the inner pentagonal circuit of the graph $G \setminus \bar{v}$, as was done by Kempe [1] and Heawood [2].

Another easily extracted result from the above proof is that whenever a plane graph can be constructed (and coloured) without employing Case B2 (Case B1 can be replaced by Case A1 to avoid subdivision), the following corollary holds:

Corollary (4LCT): *Every planar graph which can be constructed from a triangulation without subdivision of inner edges is 4-choosable.*

Acknowledgment

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